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OPTIMAL CONTROL OF SYSTEMS GOVERNED BY DELAYED-DIFFERENTIAL EQUATIONS

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Monterey, California

THESIS

OPTIMAL CONTROL OF SYSTEMS GOVERNED BY
DELAYED-DIFFERENTIAL EQUATIONS

by

Joseph Goree Hyde

Thesis Advisor: R.A. Hess

December 1972

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Optimal Control of Systems Governed by Delayed-Differential Equations

by

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Submitted in partial fulfillment of the
requirements for the degree of

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ABSTRACT

Optimal control of systems governed by delayed-differential equations is explored by using the control theory developed for systems governed by ordinary differential equations. A simple algorithm for producing a suboptimal control law with restricted feedback is presented. Two examples illustrate the computational method.

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I. INTRODUCTION

A. DELAYED-DIFFERENTIAL EQUATIONS

In many physical systems, such as rocket engines [1,2], antirolling stabilization systems for destroyers [3] and pipeline recycling systems for chemical reactors [4], the most accurate mathematical representation of system dynamics is found to be a matrix system of differential equations with finite time delay in the arguments of some of the state variables.

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + Cu(t) \qquad (1)$$

$$t \geq 0 \quad , \quad \tau \geq 0$$

$$x(t) = \phi(t) \; ; \quad -\tau \leq t \leq 0 \qquad (2)$$

Equations of this form have been known as differential-difference, hysterodifferential, functional differential, differential with transport lag, differential with deviating arguments, and delayed-differential equations, to name just a few. For the purposes of this research, equations of the form of (1) are referred to as delayed-differential equations.

B. THE OPTIMAL REGULATOR PROBLEM

The research to be described is concerned with a solution of the optimal regulator problem. Simply stated, this problem seeks the control function u*(t) (a member of the set of admissible controls u(t)) which minimizes a quadratic

cost function of the form

$$J(\phi(t),x(t),u(t)) = \int_0^{\infty} [x'(t)Qx(t) + u'(t)Ru(t)]dt > 0 \quad (3)$$

subject to the constraints of equations (1) and (2). Q is a n^2 positive semidefinite matrix; R is a ℓ^2 positive definite matrix; $\ell \leq n$.

C. NOTATION

Notation follows that established by Bryson and Ho [5]. In general, it is as follows:

1. Column vectors are denoted by lower case letters. Single subscripted lower case letters are elements of vectors.
2. Matrices are indicated by upper case letters. Doubly subscripted lower case letters represent elements of a matrix.
3. Matrix transpose is indicated by the prime symbol(').

The following are deviations from the general rules established above.

1. t represents the scalar independent variable time.
2. τ denotes the scalar time delay, possibly a function of time and state variables.
3. J is a scalar function of matrices.
4. The lower case letters i,j,ℓ,m, and n are used as indexing variables.
5. The total derivative is indicated by d.
6. A damped natural frequency is represented by ω_j.

D. PROBLEM STATEMENT

1. Introduction

Inclusion of the delay term τ in the state variable equation greatly complicates solution of the optimal

regulator problem. The well established theory of ordinary differential equation systems can not be applied with equation (1). In addition, although the necessary theory and computational procedures have been developed [6] for computation of the optimal control law, implementation of that law is often found to be either impractical or undesirable. Implementation of the optimal control law requires feedback of a continuum of states, $x(\zeta)$, $t - \tau \leq \zeta \leq t$, which are often not readily available from the system being controlled.

2. Purpose

The purpose of this research is to develop a method for approximating the optimal control law for delayed-differential systems by utilizing the optimal control formulation for ordinary differential systems. In addition, a simple algorithm is offered for restricted feedback cases.

E. BACKGROUND

1. General

Anderson and Soudack [7] indicate that the study of the delayed-differential equation appears to have started with John Bernoulli in 1728, but most presently available information has been published since 1940. Bibliographies published by R. Weiss [8] in 1959 and Chosky [9] in 1960 contain references to more than 350 separate papers dealing with the delayed-differential equation and associated problems. The annotated bibliography by R. Weiss surveys English language papers published between 1935 and 1958.

The work by Chosky is supplemental to the bibliography of Weiss and covers the period of 1916 to 1959, including reference to numerous foreign language papers.

Prior to 1960, several papers were published which addressed the problem of approximating the delay term so that the desired control law could be obtained.

In a paper published in 1940, Mason and Philbrick [10] devised a liquid level analog to a thermal process with time lag. This model was used to examine response to various types of control systems. A related work by Ziegler and Nichols [11] considered various control settings for automatic control circuits with process lag. The optimal control was determined from the area under a state time-history curve. Bretoi [12] and Minorsky [3] used first order differential equations to approximate control systems with time lags. After the control law was determined from the approximation, the actual system was adjusted to give the desired results.

In an attempt to explain why, under certain circumstances, a pendulum began to spontaneously oscillate with a frequency higher than its own damped natural frequency or that of the externally applied forcing function, Minorsky [13] expressed the system model as

$$\ddot{x}_1(t) + a_1\dot{x}_1(t) + b_1\dot{x}_1(t-\tau) + c_1x_1(t) = 0 \qquad (4)$$

Considering the time delay term τ to be small, $\dot{x}_1(t-\tau)$ was expanded in a Taylor series to yield

$$(-1)^n \frac{\tau^n}{n!} x_1^{(n+1)}(t) + \dots - \tau \ddot{x}_1(t) + \dot{x}_1(t)$$

For very large n and small τ this expression converges to a term of the form $e^{-\sigma}$. Replacing the delayed term in equation (4) by $e^{-\sigma}$ the following system model was obtained:

$$\ddot{x}_1(t) = (a_1 - b_1 e^{-\sigma})\, \dot{x}_1(t) + c_1 x_1(t) = 0$$

This equation was studied to determine regions of stability.

The major work in the theory of delayed-differential equations has been undertaken since 1960. This increasing emphasis may in part be attributed to the rising application of such equations to automatic control systems as well as the development of high speed computers.

2. Development of Theory

Bellman [14] and El'sgol'ts [15] discussed the existence and uniqueness of solutions to equations of the form of (1).

Krosovskii [16], appears to have been one of the earliest developers of an optimal control formulation for the linear quadratic problem with time delays in the states. Although he gave no explicit definitions for the parameters required to actually determine the optimal control law; the feedback was in the form of a linear combination of the states. Extension of Pontryagin's maximum principle to

systems with multiple delays was accomplished by Kharatishvii [17,18]. These works have served as a basis for numerous other papers on differential-difference equations. Orguztoreli [19], Chyung and Lee [20] have also worked with the derivation of necessary conditions for optimality in the form of the maximum principle.

Ross and Flügge-Lotz [6] developed an optimal linear control law, similar in form to Krosovskii's, for a system in which the time delay had been normalized to one. (In a later work, Ross [21] showed that a system with any number of delay terms can be transformed into a system with one delay which can be normalized to one.) This paper also presented the sufficient conditions for the existence of admissible control functions and sufficient conditions for a linear control law to be optimal. Kushner and Barnea [22] showed that a linear control law for (1) subject to a quadratic cost function in the form of (3) was optimal with respect to the class of square integrable control laws, if the solution to (1) was continuous.

3. Approximation of Delayed-Differential Equations

Because of the complexity of solving the optimal regulator problem when the system dynamics involve delayed-differential equations, considerable effort has been directed toward approximating the delayed-differential equation by systems of ordinary differential equations.

Repin [23] has shown that equation (1) can be approximated to any desired accuracy by a series of differential equations.

For systems in which the delay term is small, Jen-Wei [24] developed a procedure for replacing the delay term by a series of piece-wise linear functions which lead to a system of linear state variable equations with no time delay. Hess [25] has extended this procedure to a system in which the delay was not restricted.

Results similar to those obtained by Ross and Flügge-Lotz [6] were developed by Soliman and Ray [26,27] by approximating the delay with an arbitrarily large set of differential equations. Similar procedures have been used by Westdal and Lehn [28] for a related problem.

II. THEORY OF THE OPTIMAL CONTROL OF DELAYED-DIFFERENTIAL SYSTEMS

A. EXISTENCE AND UNIQUENESS

For an equation of the following form,

$$\dot{x}(t) = f(t,x(t),x(t-\tau))$$

$$x(t) = \phi(t) \;, \quad t_0 - \tau \leq t \leq t_0$$

$$\tau(t,x(t)) \geq 0$$

El'sgol'ts [15] has shown that if the functions f, ϕ, τ are continuous, the solution exists. If in addition

$$\left|\frac{\partial f}{\partial(x(t))}\right| \;, \quad \left|\frac{\partial f}{\partial(x(t-\tau))}\right| \;, \quad \left|\frac{d\phi(t)}{dt}\right| \;, \quad \text{and} \quad \left|\frac{d\tau}{dt}\right|$$

are bounded in the vicinity of the initial value (a Lipschitz condition in x(t)), then the solution is also unique.

Since it has been shown [6] that the optimal control, u*(t), is of the form

$$u^*(t) = -R^{-1}C'[K_0 x(t) + \int_{-\tau}^{0} K_1(\theta)x(t+\theta)d\theta]$$

the solution to the optimal regulator problem (equations 1-3) exists and is unique if the above Lipschitz condition is satisfied.

B. CONTROLLABILITY

L. Weiss [29] developed algebraic sufficient conditions for delayed-differential equations with time varying coefficients to be controllable. These conditions reduce to the ordinary algebraic criteria for controllability as the delay becomes negligible. The paper presents a method for computing the columns for the controllability matrix.

Considering a system with constant coefficients, Hewer [30] has shown that if the system in which $\tau \rightarrow 0$ is controllable, then the delayed equation is controllable for any $\tau \geq 0$. The converse of this statement is not always true. For the time invariant case, the results obtained by Weiss can be reduced to those obtained by Hewer.

C. DEVELOPMENT OF THE OPTIMAL CONTROL LAW

Ross and Flügge-Lotz [6] and Ross [21] have developed the optimal control law for equation (1) subject to (3) as follows;

1. Admissible Control

If (1) is controllable, the set of admissible control functions, $u(t)$, are those measureable functions in $L_2[0,\infty]$ which yield a finite value for (3) for any set of initial conditions $\phi(t)$.

2. Sufficient Conditions for a Linear Feedback Control Law to be Optimal

The optimal control law of (1) subject to the cost function (3) and a specified set of initial condition functions (2) is of the form

$$u^*(t) = -R^{-1}C'[K_0x(t) + \int_{-\tau}^{0} K_1(\theta)x(t+\theta)d\theta] \tag{5}$$

$$t \geq 0 \; ; \quad x(t) = \phi(t) \; , \quad -\tau \leq t \leq 0$$

if

a. $u^*(t)$ is a stable control law (admissible in this case)

b. K_0 (a symmetric positive definite matrix), $K_1(\theta)$ and $K_2(\theta,\zeta)$ satisfy the following relationships

$$A'K_0 + K_0A - K_0CR^{-1}K_0 + K_1'(0) + K_1(0) + Q = 0 \tag{6}$$

$$\frac{dK_1(\theta)}{d\theta} = (A' - K_0CR^{-1}C')K_1(\theta) + K_2(0,\theta) \qquad -\tau \leq \theta \leq 0 \tag{7}$$

$$\frac{\partial K_2(\theta,\zeta)}{\partial\theta} + \frac{\partial K_2(\theta,\zeta)}{\partial\zeta} = -K_1'(\zeta)CR^{-1}C'K_1(\theta) \tag{8}$$

$$-\tau \leq \zeta \leq 0 \; , \quad -\tau \leq \theta \leq 0$$

$$K_1(-\tau) = K_0B \tag{9}$$

$$K_2(-\tau,\theta) = B'K_1(\theta) \tag{10}$$

If $B = 0$, equivalent to a system with no delay term, the above conditions reduce to the familiar algebraic Riccati equation. The first term in $u^*(t)$ is the optimal control law for the undelayed system; the second and more complicated term accounts for the addition of the delay term.

Under the above conditions, the quadratic cost function (3) can be expressed in terms of the initial condition function (2).

$$J(\phi,u^*) = \phi'(0)K_0\phi(0) + 2\phi'(0) \int_{-\tau}^{0} K_1(\theta)\phi(\theta)d\theta$$

$$+ \int_{-\tau}^{0}\int_{-\tau}^{0} \phi'(\zeta)K_2(\theta,\zeta)\phi(\theta)d\zeta d\theta \qquad (11)$$

Although the control law, u*(t), is optimal for an arbitrary initial condition function which is continuously differentiable, the value of (3) is, as would be expected, a direct function of the initial conditions.

3. Stability of the Optimal Control Law

In order to apply the optimal control law, some approximate method of computation is required. Once the control law is calculated, there is no known algorithmic method by which the stability of the control can be assessed [6]. Determination of stability requires that the closed loop system performance be investigated for a set of initial condition functions large enough to cover those expected to be encountered by the control system. If no unstable behavior is detected, the control law can be assumed to be stable in that particular application.

III. APPROXIMATION OF THE OPTIMAL CONTROL LAW FOR DELAYED-DIFFERENTIAL SYSTEMS

A. APPROXIMATION OF DELAYED-DIFFERENTIAL EQUATIONS

Repin [23] has shown that equation (1) can be approximated by a series of equations, determined by defining a new set of state variables. The state variables are defined as follows:

$$\begin{aligned}
x_1{}^*(t) &= x(t) \\
x_2{}^*(t) &= x(t - \tfrac{\tau}{m-1}) \\
&\ \vdots \\
x_i{}^*(t) &= x(t - \tfrac{i\tau}{m-1}) \\
&\ \vdots \\
x_m{}^*(t) &= x(t - \tau)
\end{aligned} \tag{12}$$

Where m is a positive integer equal to or greater than two, $x_i{}^*(t)$ is a n×1 column vector composed of the elements of the original state vector.

Using simple forward difference formulas for estimating derivatives, the following set of equations can be developed:

$$\begin{aligned}
\dot{x}_1^*(t) &= A\,x_1^*(t) + B\,x_m^*(t) + C\,u(t) \\
\dot{x}_2^*(t) &= (x_1^*(t) - x_2^*(t))\,\frac{m-1}{\tau} \\
&\vdots \\
\dot{x}_i^*(t) &= (x_{i-1}^*(t) - x_i^*(t))\,\frac{m-1}{\tau} \\
&\vdots \\
\dot{x}_m^*(t) &= (x_{m-1}^*(t) - x_m^*(t))\,\frac{m-1}{\tau}
\end{aligned} \tag{13}$$

This series of first order, linear, matrix differential equations can be shown to be expressible as

$$\dot{y}(t) = A_0\,y(t) + C_0\,u(t) \tag{14}$$

where

$$A_0 \equiv \begin{bmatrix} A & 0 & 0 & \cdots\; B \\ \frac{(m-1)}{\tau}\,I & \frac{-(m-1)}{\tau}\,I & 0 & \cdots\; 0 \\ 0 & \frac{(m-1)}{\tau}\,I & \frac{-(m-1)}{\tau}\,I & \cdots\; 0 \\ & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots\; \frac{-(m-1)}{\tau}\,I \end{bmatrix} \tag{15}$$

$$Q_0 \equiv \begin{bmatrix} Q & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & 0 \end{bmatrix} \qquad C_0 \equiv \begin{bmatrix} C \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{16}$$

$$y(t) \equiv \begin{bmatrix} x_1^*(t) \\ x_2^*(t) \\ \vdots \\ x_i^*(t) \\ \vdots \\ x_m^*(t) \end{bmatrix} \tag{17}$$

Repin has shown that by selecting m large enough, equation (1) can be approximated to any degree of accuracy desired, and in the limit as $m \to \infty$, the representation presented above becomes exact.

B. OPTIMAL AND SUBOPTIMAL CONTROL OF ORDINARY DIFFERENTIAL SYSTEMS

For the time invariant linear system of differential equations

$$\dot{x}(t) = A_0\, x(t) + C_0\, u(t) \quad ; \quad t \geq 0 \; ; \quad x(0) = \phi$$

with a quadratic cost function in the form of (3), the optimal control law has been shown to be

$$u^*(t) = -R^{-1}\, C_0'\, V\, x(t) \tag{18}$$

where V is the symmetric positive definite solution to the algebraic Riccati equation

$$V A_0 + A_0' V - V C_0 R^{-1} C_0' V + Q_0 = 0 \tag{19}$$

When the optimal control law is applied, the value of the cost function (3) can be expressed as

$$J = \phi' V \phi$$

Elkind and Falb [31] have shown that if some control $u(t) = -G\,x(t)$ (G is a matrix of constants conformal to $x(t)$) is applied, instead of the optimal control law (18), the cost function can be expressed as

$$J_0 = \phi' V_0 \phi \geq \phi' V \phi \qquad (20)$$

further, if the eigenvalues of $(A_0 - C_0 G)$ have negative real parts, V_0 is the symmetric positive definite solution to

$$(A_0 - C_0 G)' V_0 + V_0 (A_0 - C_0 G) + G' R^{-1} G + Q_0 = 0 \qquad (21)$$

These expressions can be used to determine the value of (3) for any stable feedback composed of a linear combination of states.

C. APPROXIMATION OF THE CONTROL LAW

It has been shown that (1) can be approximated by a system of differential equations

$$\dot{y}(t) = A_0\, y(t) + C_0\, u(t) \quad ; \quad t \geq 0 \qquad (22)$$

$$y(t) = \overline{\Phi} \quad ; \quad t = 0$$

with $\dot{y}(t)$, $y(t)$, A_0, and C_0 as defined by (12-17). The initial condition matrix is

$$\overline{\phi} \equiv \begin{bmatrix} \phi(0) \\ \phi(\frac{-\tau}{m-1}) \\ \vdots \\ \phi(-\tau) \end{bmatrix} \tag{23}$$

The cost function is

$$J(y(0),y(t),u(t)) = \int_0^\infty [y'(t)\, Q_0\, y(t) + u'(t)\, R\, u(t)]\, dt \tag{24}$$

The hypothesis here is that an accurate approximation to the optimal control law (5) for the delayed-differential system (1) can be obtained from the optimal control law (18) for an ordinary differential system. This ordinary differential system is, of course (22), the one which approximates the delayed-differential system (1).

Rigorous proof of this hypothesis is difficult. Instead, the following section shows that this approximate method is exactly equivalent to the computational approximation which Ross utilizes in solving for his optimal control law for system (1).

D. EQUIVALENCE OF COMPUTATIONAL APPROACHES*

1. Ross's Method

In order to compute the optimal control law (5), Ross [21] replaces the conditional equations (6-10) by

---

*For convenience, and without loss of generality, the time delay τ has been normalized to unity in the discussion which follows.

finite difference formulas on the surface $-1 \leq \theta \leq 0$, $-1 \leq \zeta \leq 0$. The resulting equations are

$$A'K_0 + K_0 A - K_0 C R^{-1} C' K_0 + K_1'(0) + K_1(0) + Q = 0 \tag{25}$$

$$\left\{K_1\left(\frac{-(i-1)}{m-1}\right) - K_1\left(\frac{-i}{m-1}\right)\right\} \left(\frac{1}{m-1}\right) = \left\{A' - K_0 C R^{-1} C'\right\} K_1 \left(\frac{-(i-1)}{m-1}\right)$$

$$+ K_2\left(0 , \frac{-(i-1)}{m-1}\right) ,$$

$$1 \leq i \leq m-1 \tag{26}$$

$$\left\{K_2 \left(\frac{-(i-1)}{m-1} , \frac{-(j-1)}{m-1}\right) - K_2 \left(\frac{-i}{m-1} , \frac{-(j-1)}{m-1}\right)\right.$$

$$\left. + K_2 \left(\frac{-(i-1)}{m-1} , \frac{-(j-1)}{m-1}\right) - K_2 \left(\frac{-(i-1)}{m-1} , \frac{-j}{m-1}\right)\right\} \left\{\left(\frac{1}{m-1}\right)\right\}$$

$$= -K_1' \left(\frac{-(i-1)}{m-1}\right) C R^{-1} C' K_1 \left(\frac{-(j-1)}{m-1}\right)$$

$$1 \leq i \leq m-1$$

$$1 \leq j \leq m-1 \tag{27}$$

$$K_1(-1) = K_0 B \tag{28}$$

$$K_2 \left(-1 , \frac{-j}{m-1}\right) = B' K_1 \left(\frac{-j}{m-1}\right) , \quad 0 \leq j \leq m-1 \tag{29}$$

with A, B, C, Q, and R the same matrices appearing in (1) and (3).

Now for the same value of the integer m, let K be the positive semidefinite solution of the algebraic Riccati equation

$$A'_{m-1} K + K A_{m-1} - K C_{m-1} R^{-1} C'_{m-1} K + Q_{m-1} = 0 \qquad (30)$$

where

$$A_{m-1} = \begin{bmatrix} A & 0 & 0 & \cdots & B \\ (m-1)I & -(m-1)I & 0 & \cdots & 0 \\ 0 & (m-1)I & -(m-1)I & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & (m-1)I & -(m-1)I \end{bmatrix}$$

$$Q_{m-1} = \begin{bmatrix} Q & 0 & & 0 \\ 0 & 0 & \cdot & 0 \\ & \cdots & & \\ 0 & 0 & \cdots & 0 \end{bmatrix} \qquad C_{m-1} = \begin{bmatrix} C \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (31)$$

(I is the n × n identity matrix, A_{m-1} and Q_{m-1} are square matrices having n × m rows and columns, and C_{m-1} is n × m by ℓ)

If one considers the (symmetric) matrix K to consist of m^2 sub-blocks, K_{ij}, for $0 \leq i \leq m-1$, and $0 \leq j \leq m-1$, of n × n

$$K = \begin{bmatrix} K_{0,0} & K_{0,1} & \cdots & K_{0,m-1} \\ K_{1,0} & K_{1,1} & \cdots & K_{1,m-1} \\ & & \cdots & \\ K_{m-1} & & \cdots & K_{m-1,m-1} \end{bmatrix} \qquad (32)$$

then (30) becomes a set of algebraic equations in the unknowns K_{ij}.

Ross then points out that if one dissects (30) and compares those algebraic relations with the algebraic relations (25-29), the two sets of equations are found to be in direct correspondence. Specifically:

$$K_0 = K_{0,0} \qquad (33)$$

$$K_1\left(\frac{-i}{(m-1)}\right) = (m-1)\,K_{0,i+1}\;, \qquad 0 \leq i \leq m-2$$

$$K_1(-1) = K_0\,B = K_{0,0}'\,B \qquad (34)$$

$$K_2\left(\frac{-i}{(m-1)}\;,\;\frac{-j}{(m-1)}\right) = (m-1)^2\,K_{i+1,j+1}\;, \qquad 0 \leq i \leq m-2 \text{ and } 0 \leq j \leq m-2$$

$$K_2\left(-1\;,\;\frac{-j}{(m-1)}\right) = B'\,K_1\left(\frac{-j}{(m-1)}\right)\;, \qquad 0 \leq j \leq m-1 \;:$$

$$K_2\left(\frac{-i}{(m-1)}\;,\;-1\right) = K_1'\left(\frac{-i}{(m-1)}\right)B \qquad 0 \leq i \leq m-1 \qquad (35)$$

Thus, solution of the Riccati equation (30) is equivalent to solution of equations (25-29). Since efficient algorithms exist for the solution of Riccati equations of this form, Ross bases his computational approach on the solution of equation (30). Interpolation between the resulting values of K_0 , $K_1\left(\frac{-i}{m-1}\right)$ for $0 \leq i \leq m-1$ and $K_2\left(\frac{-i}{m-1}\;,\;\frac{-j}{m-1}\right)$ for $i,j = 0,1,\ldots,m-1$, yields approximations to K_0, $K_1(\theta)$ and $K_2(\zeta,\theta)$ in equations (5) and (11). The computation is

terminated when no significant improvement in the approximation of K_0, $K_1(\theta)$, $K_2(\zeta,\theta)$ occurs as m is increased.

2. The Method of Section III. C.

It has been shown that the optimal control law for (22) with a cost function of the form (24) is

$$u^*(t) = -R^{-1} C_0' V y(t) \tag{36}$$

where V is the solution of the Riccati equation

$$A_0' V + V A_0 - V C_0 R^{-1} C_0' V + Q_0 = 0 \tag{37}$$

V is a symmetric $(n \times m)^2$ matrix composed of m^2 sub-blocks V_{ij}:

$$V = \begin{bmatrix} V_{11} & \cdots & V_{1m} \\ \vdots & & \vdots \\ V_{m1} & & V_{mm} \end{bmatrix} \tag{38}$$

The value of the cost function J is given by

$$J = \phi' V \phi \tag{39}$$

Equations (36) and (39) can be rewritten

$$u^*(t) = -R^{-1} C_0' [V_{11} y_1(t) + \sum_{i=2}^{m} V_{1i}\, y_i(t)] \tag{40}$$

$$J = \phi'(0)\, V_{11}\, \phi(0) + 2\, \phi'(0) \sum_{i=2}^{m} V_{1i}\, \phi\left(\frac{1-i}{m-1}\right) + \sum_{j=2}^{m} \sum_{i=2}^{m} \phi'\left(\frac{1-i}{m-1}\right) V_{ij}\, \phi\left(\frac{1-j}{m-1}\right) \tag{41}$$

or

$$u^*(t) = -R^{-1} C_0' \left[V_{11}\, y_1(t) + \sum_{i=1}^{m-1} V_{1,i+1} \cdot (m-1) \cdot y_{i+1}(t) \cdot \frac{1}{m-1} \right] \tag{42}$$

$$J = \phi'(0)\, V_{11}\, \phi(0) + 2\phi'(0) \sum_{i=1}^{m-1} V_{1,i+1} \cdot (m-1) \cdot \phi\left(\frac{-i}{m-1}\right) \cdot \frac{1}{m-1} + \sum_{j=1}^{m-1} \sum_{i=1}^{m-1} \phi'\left(\frac{-j}{m-1}\right) V_{i+1,j+1} \cdot (m-1)^2 \cdot \phi\left(\frac{-i}{m-1}\right) \cdot \frac{1}{(m-1)^2} \tag{43}$$

Now the integrals in the optimal control law and cost function of equations (5) and (11) can be approximated by finite summations as follows:

$$u^*(t) \approx -R^{-1} C' \left[K_0\, x(t) + \sum_{i=1}^{m-1} K_1\left(\frac{-(i-1)}{m-1}\right) x\left(t - \frac{i}{m-1}\right) \cdot \frac{1}{m-1} \right] \tag{44}$$

$$J \approx \phi'(0)\, K_0\, \phi(0) + 2\phi'(0) \sum_{i=1}^{m-1} K_1\left(\frac{-(i-1)}{m-1}\right) \phi\left(\frac{-i}{m-1}\right) \cdot \frac{1}{m-1} + \sum_{j=1}^{m-1} \sum_{i=1}^{m-1} \phi'\left(\frac{-j}{m-1}\right) K_2\left(\frac{-(i-1)}{m-1}, \frac{-(j-1)}{m-1}\right) \phi\left(\frac{-i}{m-1}\right) \cdot \frac{1}{(m-1)^2} \tag{45}$$

Recalling that equations (12) and (17) state that

$$y_i(t) = x\left(t - \frac{i-1}{m-1}\right) \qquad i = 1,2,\ldots,m$$

then each term in equations (42-43) corresponds to a term in (44-45). This correspondence strongly suggests

$$V_{1,i+1} = K_1\left(\frac{-(i-1)}{m-1}\right) \frac{1}{m-1}$$

$$V_{i+1,j+1} = K_2\left(\frac{-(i-1)}{m-1}, \frac{-(j-1)}{m-1}\right) \cdot \frac{1}{(m-1)^2} \qquad (46)$$

Assuming the validity of equations (46), one can now approximate the optimal control law for delayed-differential equations by utilizing the optimal control formulation for ordinary differential systems.

3. Equivalence

The validity of equations (46) and their equivalence to Ross's solution can be proven by noting that the Riccati equation (30) which Ross utilizes to generate his solution is identical to the Riccati equation (37) used to generate the differential solution.

E. ADVANTAGES OF THE DIFFERENTIAL APPROACH

There are certain advantages to be gained by using the differential approximation scheme in calculating the optimal control law for delayed-differential systems.

First, the closed loop eigenvalues of the approximating differential system yield information about the quality of

the differential approximation, and about the stability characteristics of the actual, optimally controlled delayed-differential system. As Hess [25] points out, the largest closed loop damped natural frequency of the differential system should be small when compared to the frequency of occurrence of the station points used in defining $\dot{x}_2^*(t)$ through $\dot{x}_m^*(t)$ in (13). This means ensuring that

$$\frac{\omega_j}{2\pi(m-1)} << 1.0 \qquad j=1,2,\ldots,n\times m \qquad (47)$$

where ω_j is the magnitude of the imaginary part of the j^{th} closed loop eigenvalue. The smaller the left hand side of equation (47), the more accurate are the forward difference expressions for $\dot{x}_2^*(t)$ through $\dot{x}_m^*(t)$ in (13). If the restrictions of equation (47) are met, the closed loop eigenvalues of the differential system give excellent information regarding the stability characteristics of the actual, optimally controlled delayed-differential system.

Second, the differential approach is amenable to systems with time varying parameters. This means that one should be able to find an approximation to the optimal control law for a delayed-differential system of the form

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,x[t - \tau(t)] + C(t)\,u(t)$$

subject to a cost function of the form

$$J(\phi(t),x(t),u(t)) = \int_0^\infty [x'(t)\,Q(t)\,x(t) + u'(t)\,R(t)\,u(t)]dt$$

by utilizing the theory of optimal control of ordinary time varying differential systems.

F. ALGORITHM FOR RESTRICTED FEEDBACK

1. Introduction

The optimal control law (5) may require a digital computer for implementation. The question naturally arises as to just how well the system (1) would perform with a simpler, suboptimal control law. For example, for systems in which the time delay is relatively small, a control law of the form

$$u(t) = -G_1\,x(t) \tag{48}$$

may yield satisfactory performance, and be much easier to implement than control law (5). If law (48) is not satisfactory

$$u(t) = -G_1\,x(t) - G_2\,x(t-1) \tag{49}$$

or

$$u(t) = -G_1\,x(t) - G_2\,x(t-\tfrac{1}{2}) - G_3\,x(t-1) \tag{50}$$

might be considered. The next section offers a direct constructive procedure for determining the G_i in (48-50) based upon the optimal formulation (5).

2. Restricted Feedback

The method to be described is based upon the same linear approximations for the states of the delayed-differential

system as used in the differential approximation (13). The procedure involves representing x(t+θ) in (5) by any number of straight line segments as follows:

Zeroth approximation

$$x(t+\theta) \approx x(t) \quad , \quad 0 \leq \theta \leq -1$$

First approximation

$$x(t+\theta) \approx x(t) + [x(t) - x(t-1)]\theta \quad , \quad 0 \leq \theta \leq -1$$

Second approximation

$$x(t+\theta) \approx x(t) + [\frac{x(t) - x(t-\frac{1}{2})}{\frac{1}{2}}]\theta \quad , \quad 0 \leq \theta \leq -\tfrac{1}{2}$$

$$x(t+\theta) \approx x(t-\tfrac{1}{2}) + [\frac{x(t-\frac{1}{2}) - x(t-1)}{\frac{1}{2}}](\theta+\tfrac{1}{2}) \quad , \quad -\tfrac{1}{2} \leq \theta \leq -1$$

Third approximation

$$x(t+\theta) \approx x(t) + [\frac{x(t) - x(t-\frac{1}{3})}{\frac{1}{3}}]\theta \quad , \quad 0 \leq \theta \leq -\tfrac{1}{3}$$

$$x(t+\theta) \approx x(t-\tfrac{1}{3}) + [\frac{x(t-\frac{1}{3}) - x(t-\frac{2}{3})}{\frac{1}{3}}](\theta+\tfrac{1}{3}) \quad , \quad -\tfrac{1}{3} \leq \theta \leq -\tfrac{2}{3}$$

$$x(t+\theta) \approx x(t-\tfrac{2}{3}) + [\frac{x(t-\frac{2}{3}) - x(t-1)}{\frac{1}{3}}](\theta+\tfrac{2}{3}) \quad , \quad -\tfrac{2}{3} \leq \theta \leq -1$$

A simplified control law based upon any one of these approximations can be computed by substituting one of the above expressions for $x(t+\theta)$ into the integral expression in the optimal control law (5). For example, using the second approximation

$$u^*(t) \approx -R^{-1} C' [K_0 x(t) + \int_{-\frac{1}{2}}^{0} K_1(\theta)\{x(t) + 2[x(t) - x(t-\tfrac{1}{2})]\theta\} \, d\theta$$

$$+ \int_{-1}^{-\frac{1}{2}} K_1(\theta)\{x(t-\tfrac{1}{2}) + 2[x(t-\tfrac{1}{2}) - x(t-1)](\theta+\tfrac{1}{2})\} d\theta]$$

which yields

$$u^*(t) \approx -R^{-1} C' \Big\{[K_0 + \int_{-\frac{1}{2}}^{0} (1 + 2\theta) K_1(\theta) \, d\theta] \, x(t)$$

$$+ [-\int_{-\frac{1}{2}}^{0} 2\theta \; K_1(\theta) \, d\theta + \int_{-1}^{-\frac{1}{2}} (2 + 2\theta) K_1(\theta) \, d\theta] \, x(t-\tfrac{1}{2})$$

$$+ [-\int_{-1}^{-\frac{1}{2}} (1 + 2\theta) K_1(\theta) \, d\theta] \; x(t-1)\Big\}$$

$$\approx -G_1 \, x(t) - G_2 \, x(t-\tfrac{1}{2}) - G_3 \, x(t-1)$$

Since K_0 and $K_1(\theta)$ have already been calculated, the expressions above can be numerically integrated to yield a simplified, suboptimal control law.

The performance of the delayed-differential system with any of these suboptimal control laws depends, of course, upon how well the straight line approximations represent

$x(t+\theta)$. In this light, if a suboptimal law based upon the zeroth approximation yields a significantly larger value of the cost function than the optimal law, a higher order approximation (rather than a different value of G_1) is needed. The large increment in the cost function is indicating that

$$x(t+\theta) \approx x(t) \ ; \qquad 0 \leq \theta \leq -1$$

is not a satisfactory approximation.

3. Stability and Cost Function Evaluation

Information about the stability of the suboptimal control law can be obtained from the closed-loop eigenvalues of the differential system as outlined in Section III. E. The cost function can be evaluated either directly from simulation or in approximate fashion by using expressions similar to (20-21).

IV. SAMPLE CONTROLLER DEVELOPMENT

A. SYSTEMS TO BE STUDIED

The scalar equations

$$\dot{x}(t) = a_{11}\, x(t) + b_{11} x(t-1) + c_{11}\, u(t) \tag{51}$$

$$x(t) = 1 + t \ , \quad -1 \leq t \leq 0 \tag{52}$$

$$J = \int_0^\infty (x^2(t) + u^2(t))dt \tag{53}$$

were studied for $a_{11} = b_{11} = -c_{11} = -1$ (Case 1) and $a_{11} = b_{11} = -c_{11} = -2$ (Case 2).

For Case 1, the required order of the differential approximation was determined to be ten; for Case 2, the order was determined to be fourteen. These values for m insured that $\omega_j/(2\pi(m-1)) \approx .1$, $j=1,2,\ldots,m$

The differential approximations, expressed in matrix notation are:

$$\dot{y}(t) = A_0\, y(t) + C_0\, u(t) \quad , \quad t \geq 0 \tag{54}$$

$$y(0) = \overline{\phi} \tag{55}$$

$$J = \int_0^\infty (y'(t)\, Q_0\, y(t) + u^2(t))dt \tag{56}$$

$$A_0 \equiv \begin{bmatrix} a_{11} & 0 & \cdots & 0 & b_{11} \\ (m-1) & -(m-1) & \cdots & 0 & 0 \\ 0 & (m-1) & \cdots & 0 & 0 \\ & & \vdots & & \\ 0 & 0 & & (m-1) & -(m-1) \end{bmatrix} \tag{57}$$

$$C_0' = [c_{11} \quad 0 \quad . \; . \; . \quad 0 \quad 0]$$

$$\overline{\phi}' = [1 \quad (m-2)/(m-1) \quad . \; . \; . \quad (m-1-i)/(m-1) \quad . \; . \; . \quad 0] ,$$

$$Q_0 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & \cdot & \\ 0 & 0 & & 0 \end{bmatrix}$$

B. FORMULATION OF THE OPTIMAL CONTROL LAW

The optimal control law for the delayed-differential system (51-53) was obtained by utilizing the differential approximation (54-56). The matrix Riccati equation (19) for the differential approximation was solved and the resulting V_{ij} used to determine K_0 and $K_1(\theta)$ via

$$K_0 = V_{11}$$

$$K_1\left(\frac{-i}{m-1}\right) = (m-1)\, V_{1,i+2} \qquad i=0,1,2,\ldots,(m-2)$$

$K_1(-1)$, not normally available from the differential approach, was obtained from Ross's solution (equation (34)) as

$$K_1(-1) = b_{11}\, K_0$$

The resulting K_0, $K_1(\theta)$ and cost function values are shown in Figures 1 and 2 for Cases 1 and 2 respectively. Figure 3

shows x(t) for the uncontrolled and optimally controlled delayed-differential system for Case 1. Figure 4 presents the optimal control function, u*(t), for Case 1. Figures 5 and 6 show similar results for Case 2.

C. FORMULATION OF A RESTRICTED FEEDBACK CONTROL LAW

Suboptimal control laws were developed for feedback of a restricted set of state variables associated with the differential approximation (51-57) by the method presented in III. F.2. These control laws are:

Zeroth approximation

Case 1

$$u(t) = -.216\,x(t)$$

Case 2

$$u(t) = -.196\,x(t)$$

First approximation

Case 1

$$u(t) = -.340\,x(t) + .123\,x(t-1)$$

Case 2

$$u(t) = -.409\,x(t) + .213\,x(t-1)$$

Second approximation

Case 1

$$u(t) = -.384\,x(t) + .0887\,x(t-\tfrac{1}{2}) + .079\,x(t-1)$$

Case 2

$$u(t) = -.4478\,x(t) + .115\,x(t-\tfrac{1}{2}) + .155\,x(t-1)$$

Third approximation

Case 1

$$u(t) = -.392\, x(t) + .0384\, x(t-.33) + .0803\, x(t-.67) + .0570\, x(t-1)$$

Case 2

$$u(t) = -.445\, x(t) + .024\, x(t-.33) + .128\, x(t-.67) + .115\, x(t-1)$$

The value of the cost function vs. the number of states in the restricted feedback is presented in Figures 7 and 8 for Cases 1 and 2, respectively. Figure 9 presents x(t) for the delayed-differential system of Case 1 with no control and with the control law determined from the third approximation above. The control function for the third approximation is shown in Figure 10. Figures 11 and 12 present similar results for Case 2. In both cases, the use of a fourth state in the feedback (third approximation) produced less than a one percent improvement in the value of the cost function.

In order to compare the restricted feedback gains obtained by the algorithm of Section III.F.2 with optimal gains associated with feedback composed of one and two states, a gradient optimization technique was mechanized. In this technique, equations (20) and (21) were solved recursively to yield the optimum values of the restricted feedback gains for the differential systems of Cases 1 and 2.*

---

*A more efficient method for accomplishing this optimization can be found in Ref. 32.

A check was then made to ensure that these gains were the optimal restricted feedback gains for the actual delayed-differential system by direct integration of equations (51) and (53). The results are

Case 1

$$u(t) = -.300\,x(t)$$

$$J = .39240$$

Case 2

$$u(t) = -.500\,x(t)$$

$$J = .3729$$

Case 1

$$u(t) = -.330\,x(t) + .130\,x(t-1)$$

$$J = .37764$$

Case 2

$$u(t) = -.400\,x(t) + .225\,x(t-1)$$

$$J = .3314$$

Comparison of these results with those obtained from the zeroth and first order approximations indicate that the algorithm was moving the restricted feedback gains in the right direction as the order of the approximations was increased. The utility of the algorithm is also borne out by Figures 7 and 8.

V. CONCLUSIONS

This research has shown that the optimal control law for a delayed-differential system can be determined to any desired accuracy from the optimal control law for an ordinary differential system. Since the theory associated with differential systems is well established, and efficient algorithms for the required computations are readily available, this approach has obvious merit.

Further, it has been demonstrated that a simple algorithm can be used to determine a suboptimal, restricted feedback control law.

The closed loop eigenvalues of the differential approximation can be used to yield information about the convergence of the approximation. Although these eigenvalues completely determine the stability of the differential approximation, they can only infer, albeit strongly, that the resulting control law will produce a stable delayed-differential system. Further research is required to ascertain the conditions under which a control law determined by this method will definitely produce a stable delayed-differential system.

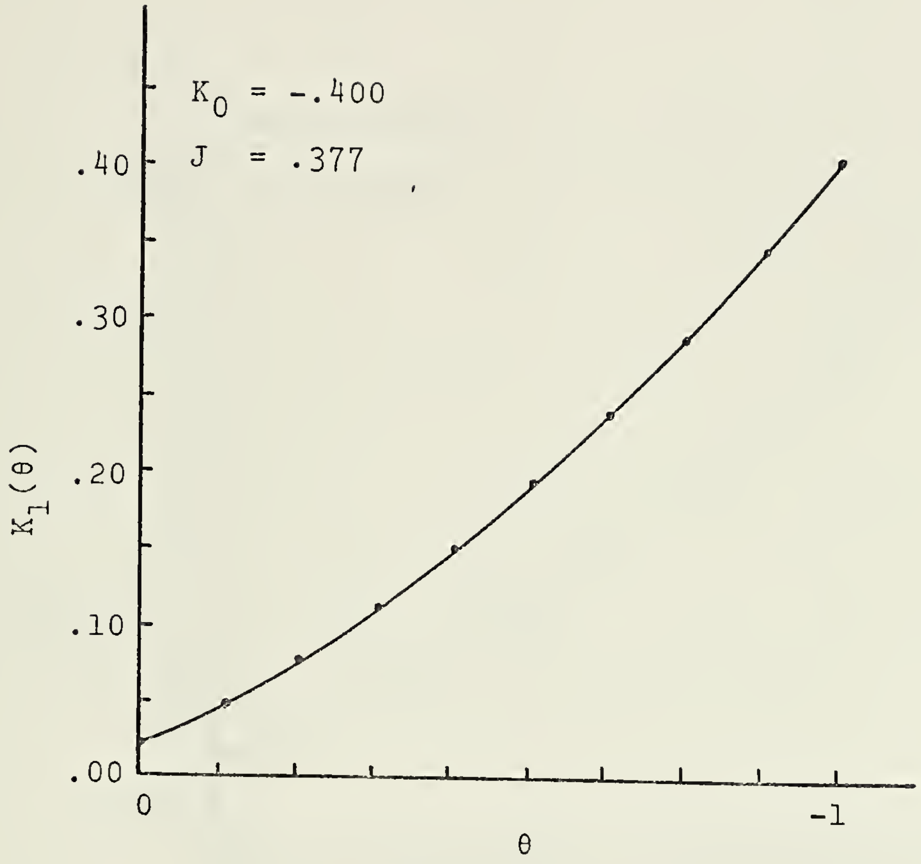

FIGURE 1

$K_1(\theta)$ vs θ, Case 1

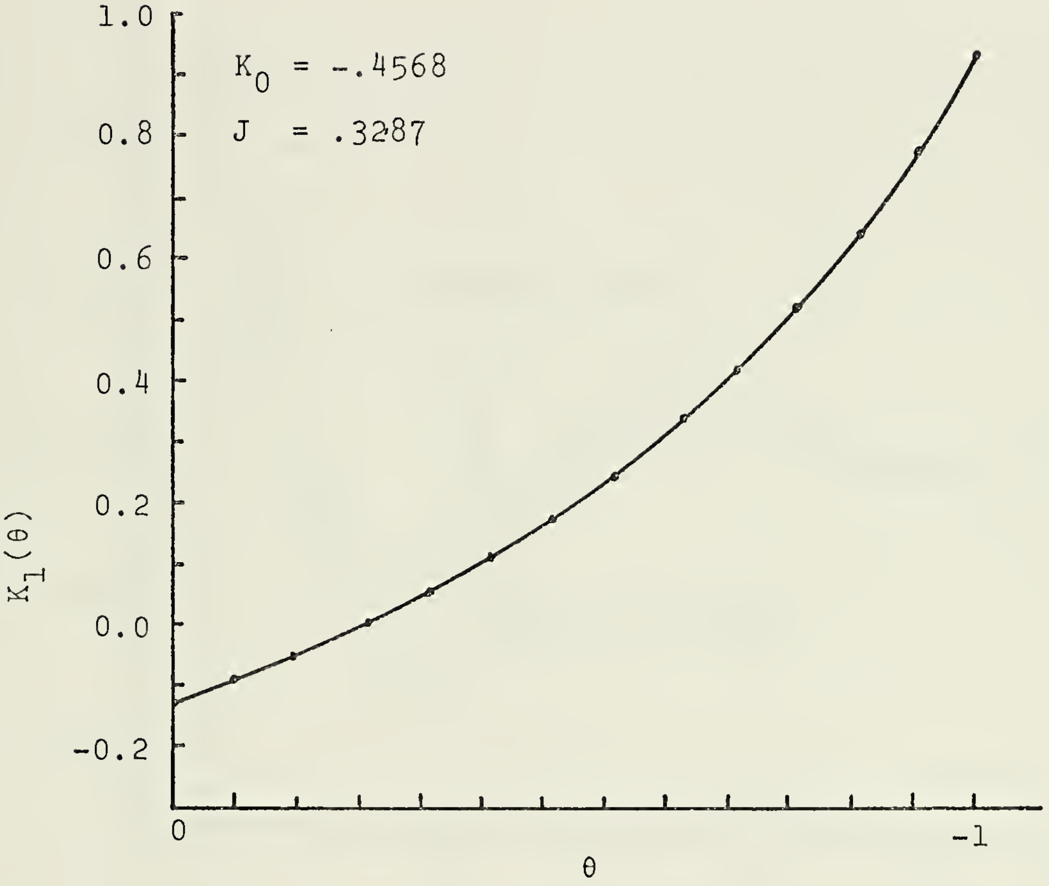

FIGURE 2

$K_1(\theta)$ vs θ, Case 2

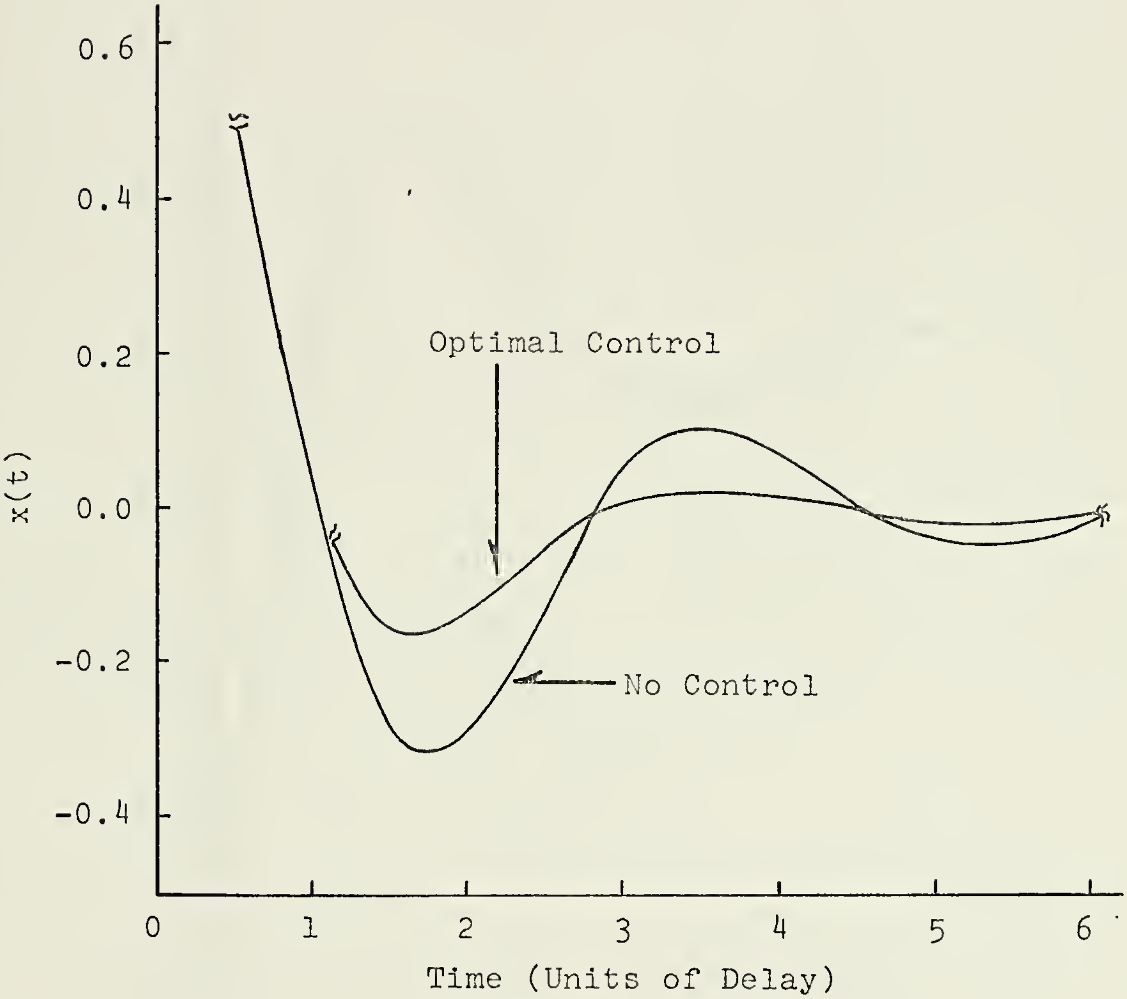

FIGURE 3

System Response With No Control and With Optimal Control, Case 1

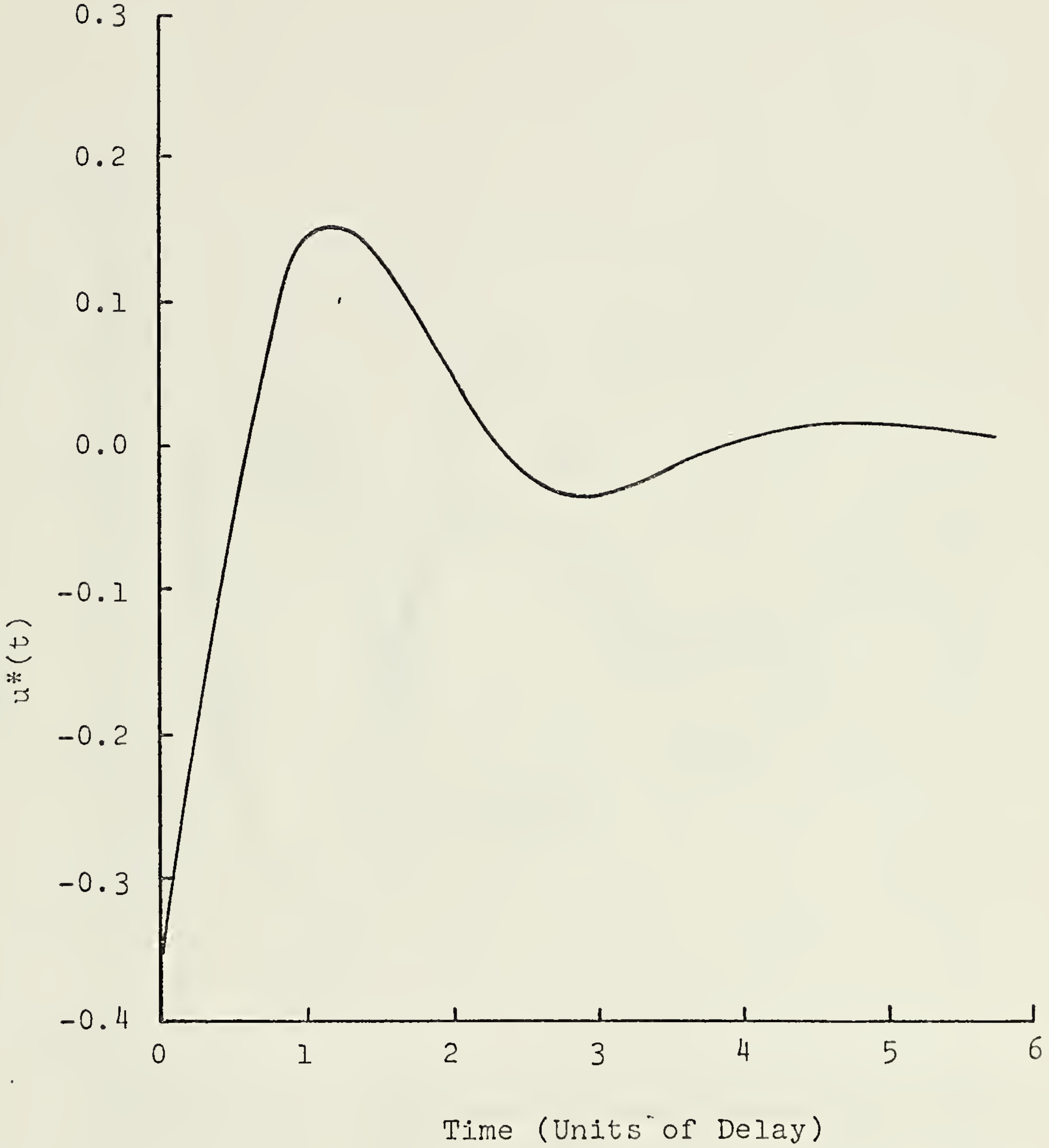

FIGURE 4

Optimal Control Function, Case 1

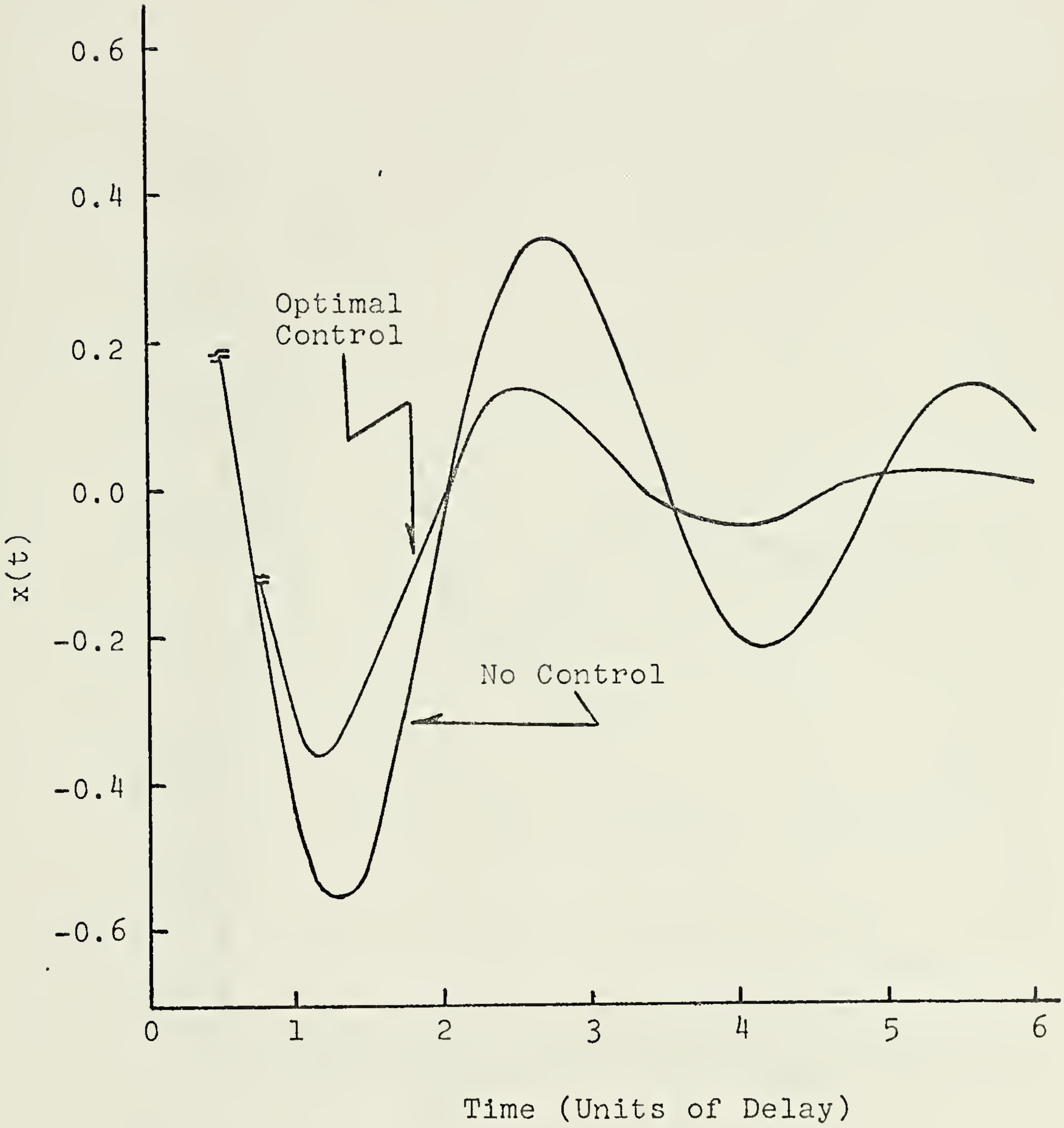

FIGURE 5

System Response With No Control and With Optimal Control, Case 2

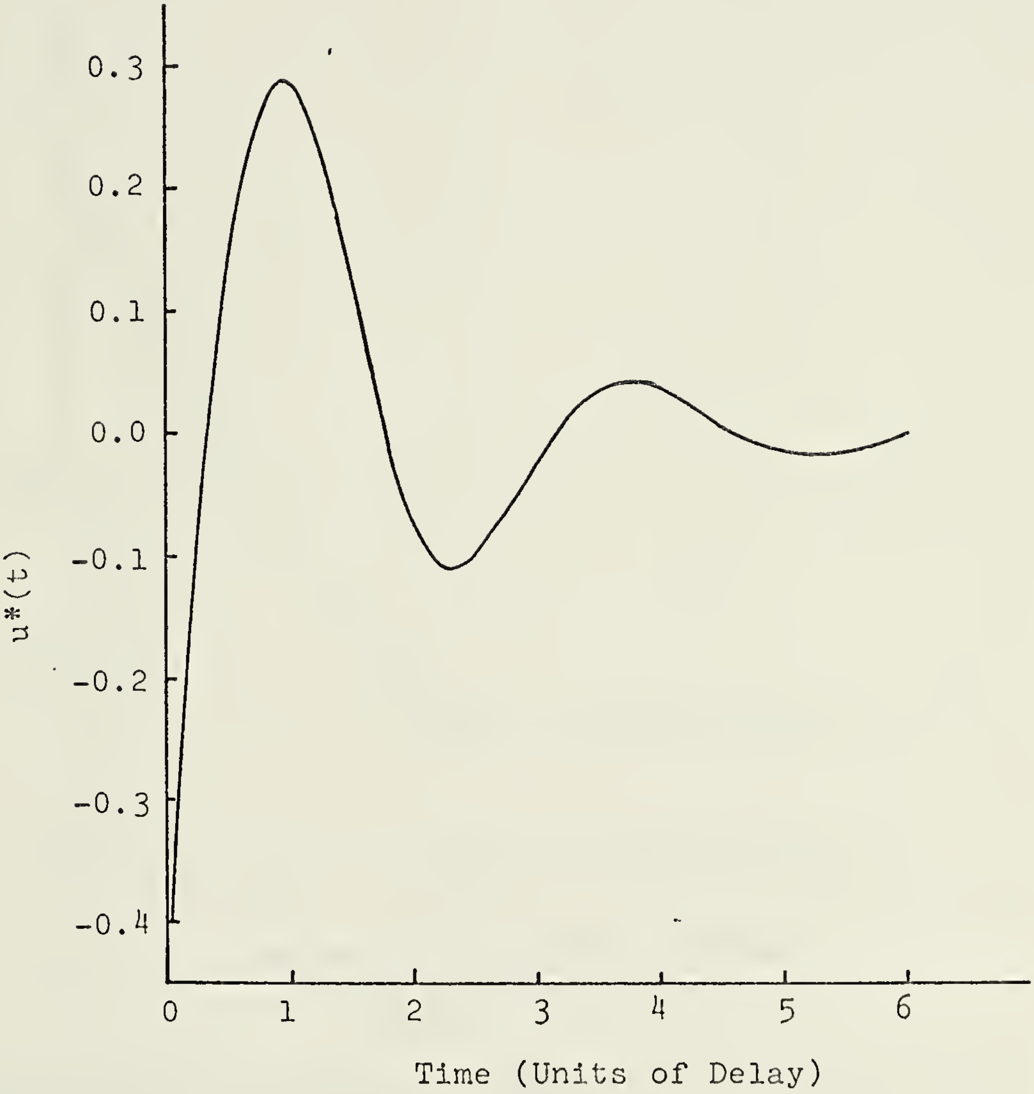

FIGURE 6

Optimal Control Function, Case 2

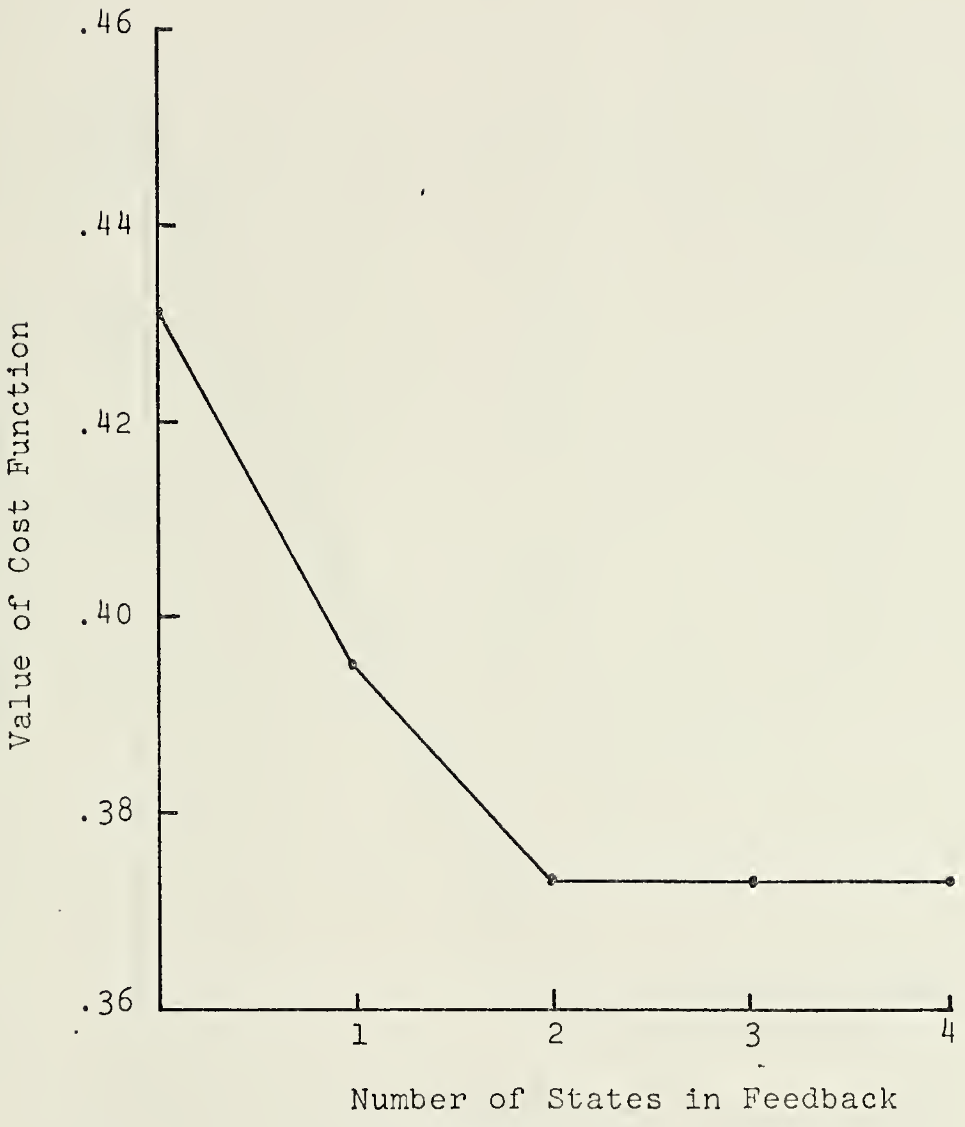

FIGURE 7

Value of Cost Function vs. Number of States in Restricted Feedback, Case 1

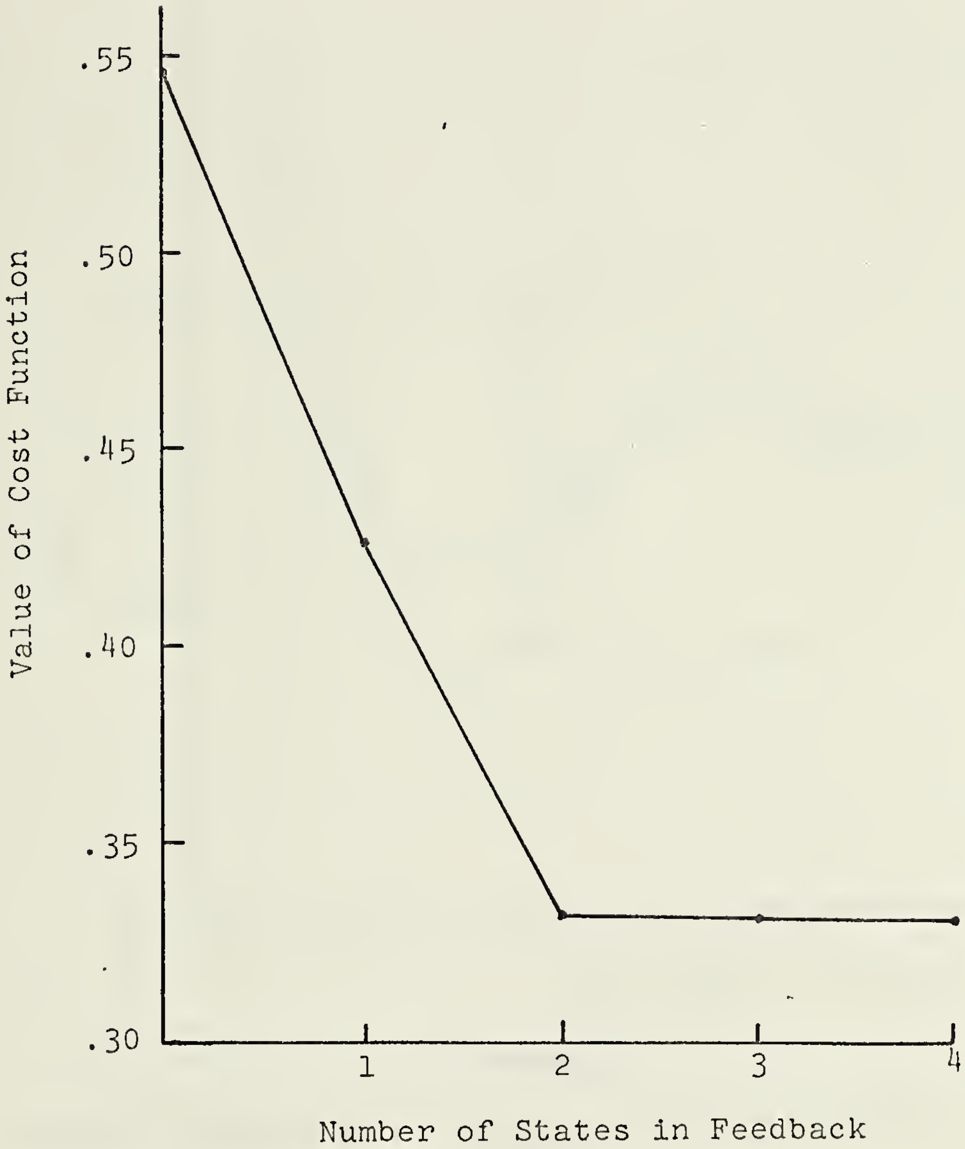

FIGURE 8

Value of Cost Function vs. Number of States in Restricted Feedback, Case 2

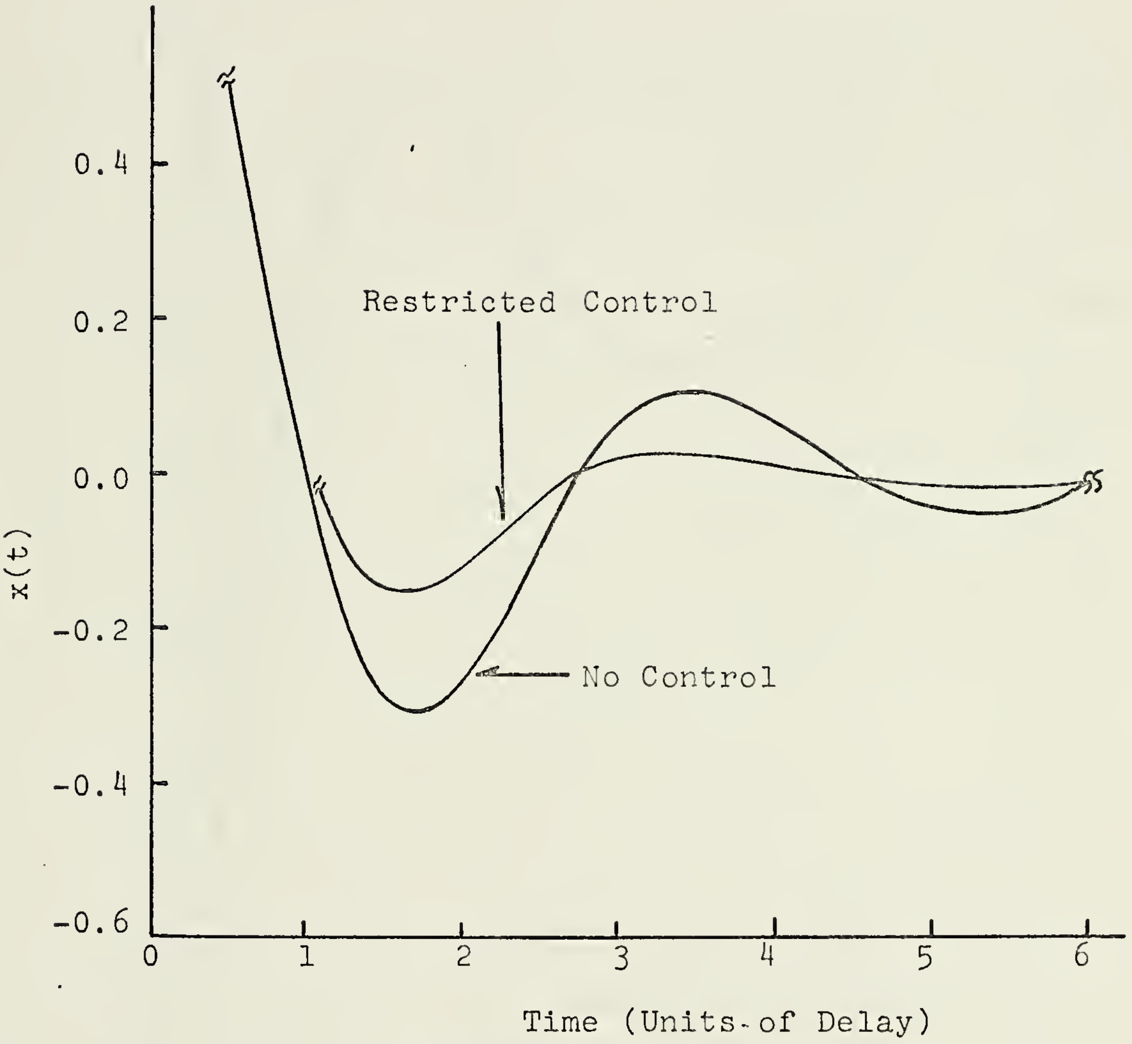

FIGURE 9

System Response With No Control and With Restricted Feedback
Case 1

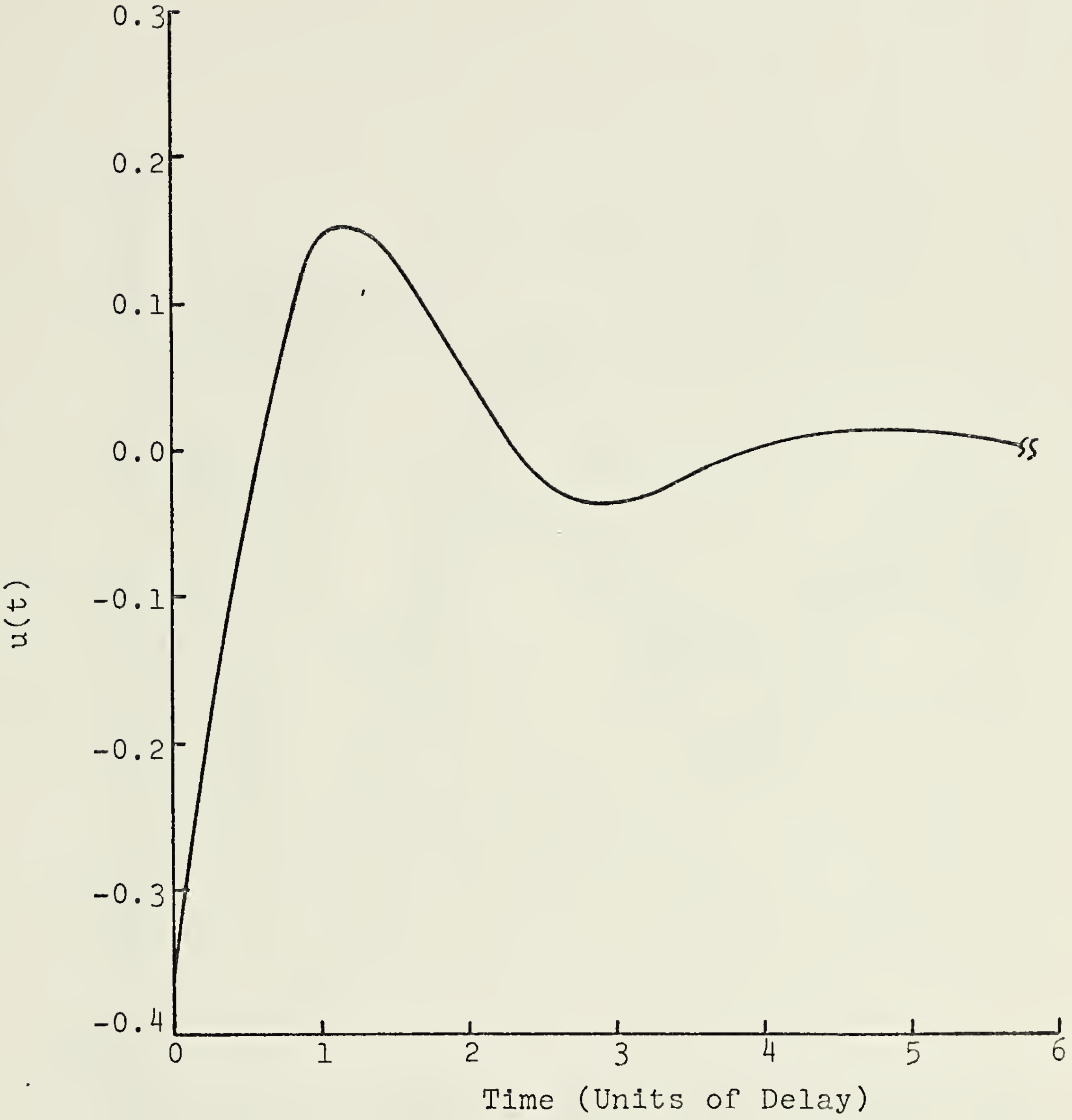

FIGURE 10
Restricted Feedback Control Function (Third Approximation)
Case 1

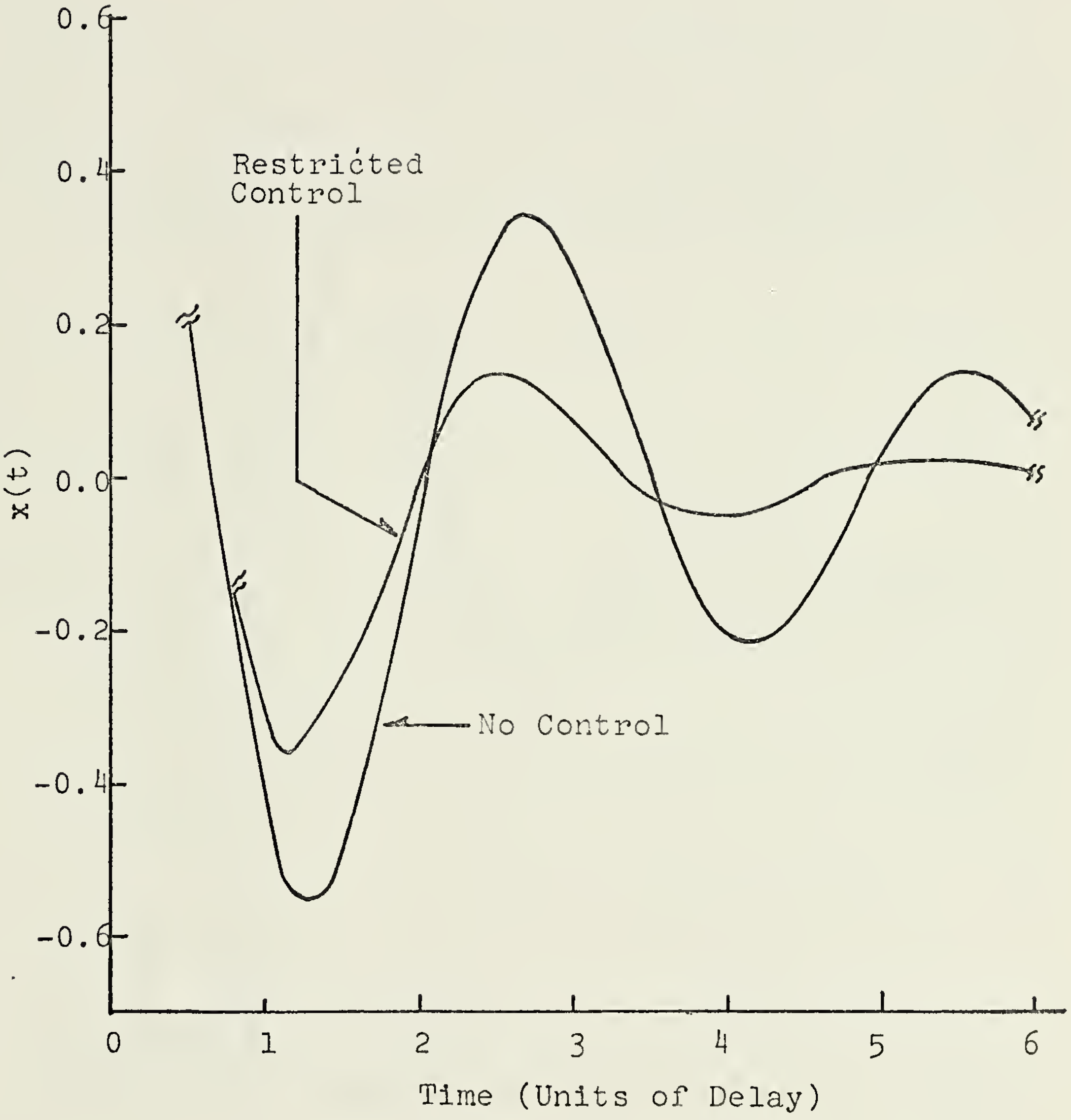

FIGURE 11

System Response With No Control and With Restricted Feedback Control, Case 2

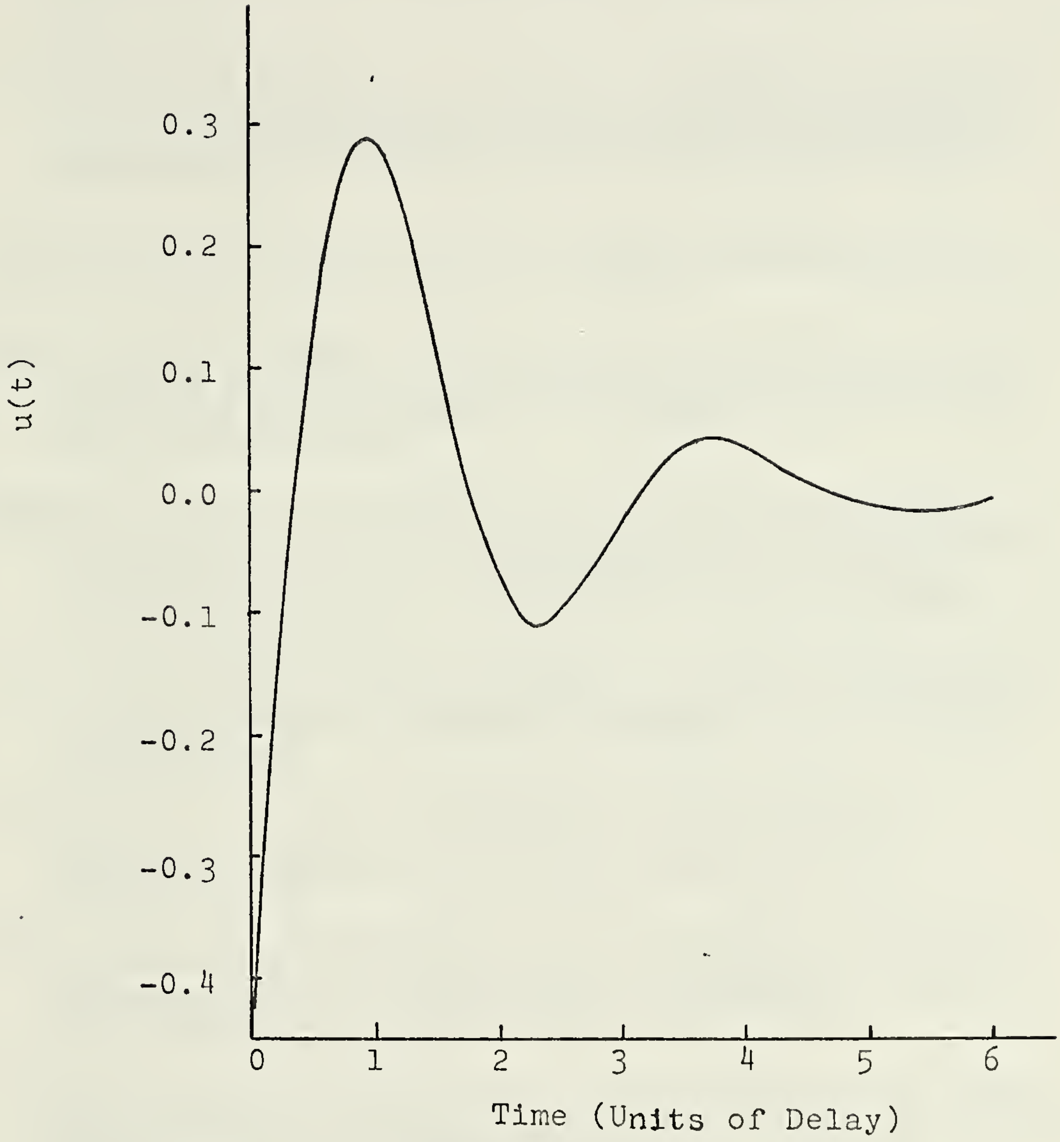

FIGURE 12

Restricted Feedback Control Function (Third Approximation)
Case 2

LIST OF REFERENCES

1. Tsien, H.S., "Linear Systems with Time Lag," *Engineering Cybernetics*, Ch. 8, McGraw-Hill, 1954.

2. Coordinated Science Laboratory Report R-530, *Optimization of Systems with Constant Pure Time Delay*, by H.C. Chan, Pp. 72-76, September 1971.

3. Minorsky, N., "Experiments with Activated Tanks," *Transactions of The American Society of Mechanical Engineers,* v. 69, Pp. 735-747, October 1947.

4. Williams, T.J., and Otto, R.E., "A Generalized Chemical Processing Model for the Investigation of Computer Control," *Transactions of The American Society of Electrical and Electronic Engineers* (Communication Electronics), v. 79, Pp. 458-473, November 1960.

5. Bryson, A.E., Jr. and Ho, Y., *Applied Optimal Control*, Pp. 438-454, Ginn and Co., 1959.

6. Ross, D.W. and Flügge-Lotz, "An Optimal Control Problem for Systems with Differential-difference Equation Dynamics," *SIAM Journal on Control*, v. 7, Pp. 609-623, November 1959.

7. Anderson, R. A. and Soudack, A.C., "On the Resonance Properties of Quasilinear Second Order Differential-Difference Equations," *International Journal of Automatic Control*, v. 11, P. 1897, January-June 1970.

8. Weiss, R., "Transportation Lag-An Annotated Bibliography," *Institute of Radio Engineers,Transactions on Automatic Control*, Pp. 56-64, May 1959.

9. Choksy, N. H., "Time Lag Systems-A Bibliography," *Institute of Radio Engineers, Transactions on Automatic Control*, Pp. 66-70, January 1970.

10. Mason, C.E. and Philbrick, G.A., "Automatic Control in the Presence of Process Lags," *Transactions of the American Society of Mechanical Engineers*, v. 67, Pp. 295-308, May 1940.

11. Ziegler, J.G. and Nichols, N.B., "Process Lags in Automatic Circuits," *Transactions of the American Society of Mechanical Engineers*, v. 65, Pp. 433-444, July 1943.

12. Bretoi, R.N., "Automatic Flight Control-Analysis of Lateral Control Problem," *Transactions of the American Society of Mechanical Engineers*, v. 74, Pp. 415-430, April 1952.

13. Minorsky, N., "Self Excited Oscillations in Dynamical Systems Possessing Retarded Action," *Journal of Applied Mechanics*, v. 9, Pp. A-65-71, June 1942.

14. Bellman, R. and Cooke, K.L., *Differential-Difference Equations*, Pp. 49-51, Academic Press, 1963.

15. El'sgol'ts, L.E., *Introduction to the Theory of Differential Equations with Deviating Arguments*, Pp. 5-19, Holden-Day, Inc., 1966.

16. Krasovskii, N.N., "Optimal Processes in Systems with Time Lag," *Proceedings of the Second Congress of the International Federation of Automatic Control*, v. 2, Pp. 327-332, 1964.

17. Pontryagin, L.S. and Others, *The Mathematical Theory of Optimal Processes*, Pp. 213-226, Interscience, 1962.

18. Kharatishvili, G.L., "A Maximum Principle in Extremal Problems with Delays," *Mathematical Theory of Control*, Balakrishnan, A.V. and others, Pp.26-34, Academic Press, 1967.

19. Oguztoreli, M.N., *Time Lag Control Systems*, Academic Press, 1966.

20. Chyung, D.H. and Lee, E.B., "Delayed Action Control Problems," *AUTOMATICA*, v. 6, Pp. 395-400, May 1970.

21. Ross, D.W., "Controller Design for Time Lag Systems via a Quadratic Criterion," *IEEE Transactions on Automatic Control*, v. AC 16, Pp. 664-672, December 1971.

22. Kushner, H.J. and Barnea, D.I., "On the Control of a Linear Functional-Differential Equation with Quadratic Cost," *SIAM Journal of Control*, v. 8, Pp. 257-272, May 1970.

23. Repin, I.M., "On the Approximate Replacement of Systems with Lag by Ordinary Dynamical Systems," *Journal of Applied Mathematics and Mechanics*, v. 29, Pp. 254-264, 1965.

24. Chen Jen-Wêi, "The Problem of Synthesizing an Optimal Controller in Systems With Time Delay," *Automation and Remote Control*, v. 23, Pp. 121-125, February 1962.

25. Hess, R., Optimal Control Approximations for Time Delay Systems, to be published in AIAA Journal 1972.

26. Soliman, M.A. and Ray, W.H., "Optimal Control of Multivariable Systems With Pure Time Delays," AUTOMATICA, v. 7, Pp. 681-689, 1971.

27. Soliman, M.A. and Ray, W.H., "Optimal Feedback Control for Linear-Quadratic Systems Having Time Delays," International Journal of Control, v. 15, Pp. 609-627, 1972.

28. Westdal, J.A.S. and Lehn, W.H., "Time Optimal Control of Linear Systems With Delay," International Journal of Control, v. 11, Pp. 599-610, 1970.

29. Weiss, L., "An Algebraic Criterion for Controllability of Linear Systems with Time Delay," IEEE Transactions on Automatic Control, v. AC-15, Pp. 443-444, August 1970.

30. Hewer, G.A., "A Note on Controllability of Linear Systems with Time Delay," IEEE Transactions on Automatic Control, v. AC-17, Pp. 733-734, October 1972.

31. Air Force Flight Dynamics Laboratory Report AFFDL-TR-187, An Optimal Control Method for Predicting Control Characteristics and Display Requirements of Manned-Vehicle Systems, by Elkind, J.I. and others, Pp. 122-125, June 1968.

32. Levine, W.S., Johnson, T.L., and Athans, M., "Optimal Limited State Variable Feedback Controllers for Linear Systems," IEEE Transactions on Automatic Control, v. AC-16, Pp. 785-798, December 1971.

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ABSTRACT

Optimal control of systems governed by delayed-differential equations is explored by using the control theory developed for systems governed by ordinary differential equations. A simple algorithm for producing a suboptimal control law with restricted feedback is presented. Two examples illustrate the computational method.

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Optimal Control						
Delayed-Differential Equations						
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